Definitions: Probability

Philosophical Logic 2025/2026

1 Basic probabilistic notions

Definition 1.1 (Probability space). *A* probability space *is a triple* $(\Omega, \mathcal{F}, \Pr)$ *where:*

1. $\Omega \neq \emptyset$ *is a non-empty set, the* sample space*;*

2. $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ *is a collection of subsets of* Ω*, the* events*, such that:*

   (a) *if* $A \in \mathcal{F}$*, then* $\Omega \setminus A \in \mathcal{F}$ *(closed under complements);*

   (b) *if* $A, B \in \mathcal{F}$*, then* $A \cup B \in \mathcal{F}$ *(closed under finite unions);*

3. $\Pr : \mathcal{F} \to [0, 1]$ *is a function, the* probability measure*, such that:*

   (a) $\Pr(\Omega) = 1$*;*

   (b) *if* $A, B \in \mathcal{F}$ *and* $A \cap B = \emptyset$*, then* $\Pr(A \cup B) = \Pr(A) + \Pr(B)$*.*

Definition 1.2 (Conditional probability). *Let* $(\Omega, \mathcal{F}, \Pr)$ *be a probability space and let* $A, B \in \mathcal{F}$ *with* $\Pr(B) > 0$*. The* conditional probability *of* A *given* B *is*

$$\Pr(A \mid B) := \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Definition 1.3 (Chain rule). *Let* $(\Omega, \mathcal{F}, \Pr)$ *be a probability space and let* $A_1, \dots, A_n \in \mathcal{F}$ *be events such that all the conditional probabilities below are defined. The* chain rule *(or* product rule*) states that*

$$\Pr(A_1 \cap \cdots \cap A_n) = \Pr(A_1) \Pr(A_2 \mid A_1) \Pr(A_3 \mid A_1 \cap A_2) \cdots \Pr(A_n \mid A_1 \cap \cdots \cap A_{n-1}).$$

For $n = 2$ *this reduces to*

$$\Pr(A \cap B) = \Pr(A) \Pr(B \mid A),$$

whenever $\Pr(A) > 0$*.*

2 Probabilities on a propositional language

Definition 2.1 (Probability function on a propositional language). *Let* $\mathcal{L}$ *be a propositional language (closed under* $\neg, \wedge, \vee$*), and let* $\vDash$ *denote classical consequence. A* probability function *on* $\mathcal{L}$ *is a function* $P : \mathcal{L} \to \mathbb{R}$ *such that, for all* $\varphi, \psi \in \mathcal{L}$*:*

1. $P(\varphi) \geq 0$;

2. *if* $\vDash \varphi$ *(i.e.* φ *is a tautology), then* $P(\varphi) = 1$;

3. *if* $\vDash \neg(\varphi \wedge \psi)$ *(i.e.* φ *and* ψ *are mutually exclusive), then* $P(\varphi \vee \psi) = P(\varphi) + P(\psi)$.

Definition 2.2 (Uncertainty). *Let* P *be a probability function on* $\mathcal{L}$. *For any sentence* $\varphi \in \mathcal{L}$ *the* uncertainty *(or* risk of error*) of* φ *relative to* P *is*

$$U_P(\varphi) := P(\neg\varphi) = 1 - P(\varphi).$$

3 Probabilistic consequence relations

Definition 3.1 (Probabilistic entailment $\vDash_P$). *Let* Γ *be a finite set of sentences of* $\mathcal{L}$ *and let* $\varphi \in \mathcal{L}$. *We say that* Γ *probabilistically entails* φ *and write*

$$\Gamma \vDash_P \varphi$$

iff for every probability function P *on* $\mathcal{L}$,

$$U_P(\varphi) \leq \sum_{\gamma \in \Gamma} U_P(\gamma).$$

Definition 3.2 (Material conditional). *For sentences* φ, ψ *in the propositional language, the* material conditional $\varphi \supset \psi$ *is the abbreviation* $\varphi \supset \psi := \neg\varphi \vee \psi$.

Theorem 3.1 (Classical fragment (no conditionals)). *Let* $\mathcal{L}_0$ *be a propositional language* without *the connective* $\rightarrow$, *closed under* $\neg, \wedge, \vee$, *and let* $\vDash_{CL}$ *denote classical entailment on* $\mathcal{L}_0$. *Then for every finite* $\Gamma \subseteq \mathcal{L}_0$ *and every* $\varphi \in \mathcal{L}_0$:

$$\Gamma \vDash_{CL} \varphi \iff \Gamma \vDash_P \varphi,$$

4 Conditionals and probabilities

Theorem 4.1 (Conditional fragment and System **P**). *Let* $\mathcal{L}_0$ *be as above and let* $\mathcal{L}^{\rightarrow}$ *be the language obtained by adding to* $\mathcal{L}_0$ *a binary connective* $\rightarrow$. *Consider the fragment of* $\mathcal{L}^{\rightarrow}$ *consisting of* non-embedded *indicative conditionals of the form* $\varphi \rightarrow \psi$ *with* $\varphi, \psi \in \mathcal{L}_0$, *and let* P *range over probability functions on* $\mathcal{L}^{\rightarrow}$ *satisfying Adams' thesis*

$$P(\varphi \rightarrow \psi) = P(\psi \mid \varphi)$$

whenever $P(\varphi) > 0$.

Let Γ *be a finite set of such non-embedded conditionals and let* α *be such a conditional. Then the consequence relation between conditionals induced by probabilistic entailment* $\vDash_P$,

$$\Gamma \vDash_P \alpha,$$

coincides with System **P**: $\Gamma \vDash_P \alpha$ *holds if and only if* α *is derivable from* Γ *using exactly the axioms and rules of System* **P**.